On the Dubrovin equations for the finite-gap potentials

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Until recently, the following question has not been considered: how to generalise constructions, related to the Dubrovin's equations (DE) in finite-gap potential theory for the Schrödinger operator, into arbitrary spectral problems. We have in mind the equations of the zeros of the Ψ-function and the trace formulas. In spite of physical interpretation of these objects as analogs of the scattering data, the generalizations are not clear, or may even be absent. This problem has an independent interest. For example, the well known Novikov's equations, appearing in a general theory of finite-gap integration, are (to all appearences) completely integrable finite-dimensional dynamical systems. A reduction of the DE to the Jacobi inverse problem demonstates the Liouville's integrability of these equations. Note that if an algebraic curve is a cover over an elliptic curve, then one can use the trace formulas to produce solutions in elliptic functions, new finite-gap potentials and some applications to nonlinear integrable partial differential equations.

In a recent note [2] an universal feature of the finite-gap potentials was revealed: they form a class, which admits an integration of the spectral problem by quadratures. It was shown there how to obtain all the ingredients of the straight spectral problem: the Ψ-formula, algebraic curve, Novikov's equations and their integrals. On the other hand, as soon as Ψ is known, it is natural to expect that the equations for its zeroes $\gamma_k(x)$ may be written using simple arguments. This can be done, and we show algorithmically how to solve the problem with the appearance of related objects: trace formulas and the Abel transformation. We do not discuss here a separate question about an exact (or one-to-one) correspondence between the following two constructions:

- an algebraic curve and the divisor of the zeroes $\{\gamma_k\}$;
- boundary conditions of the Dirichlet type $\Psi(x_{\rm o}) = \Psi(x_{\rm o}+\Omega) = 0$;

if to consider these topics in the context of recovering the potential from scattering data on an interval for arbitrary spectral problems. The set $\{\gamma_k\}$ is considered as a starting point. We demonstrate our approach on an example of a scalar operator λ-pencil of the third order.

$$\Psi''' + u(\lambda;x)\,\Psi' + v(\lambda;x)\,\Psi = 0, \tag{1}$$

where u, v are rational functions of λ with poles independent of x. We call their potential $[\,U\,]$ and fix it to be a finite-gap potential. Thus, Ψ can become the multi-point Baker–Akhiezer function [3]. An operator pencil, commuting with equation (1) (that is, the second equation on the common eigenfunction) in general has the form:

$$A(\lambda;x)\,\Psi'' + B(\lambda;x)\,\Psi' + C(\lambda;x)\,\Psi = \mu\,\Psi, \qquad C = \frac{2}{3}\,u\,A - \frac{1}{3}\,A'' - B',$$

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where A, B are polynomials in λ and differential polynomials in $[\,U\,]$. Following [2], one obtains an algebraic curve $W(\mu,\lambda)=0$ and two necessary representations for the Ψ-function.

$$\frac{\Psi'}{\Psi}=\frac{(A'+B)\,(\mu-C)+A\,C'-v\,A^2}{A\,\mu-F}=-3\,\frac{(\mu^2+Q)\,A^2+A\,F\,\mu+F^2}{A\,\Pi}+\frac{A'+B}{A}, \quad (2)$$

$$\begin{gathered}-3\,F=A\,A''+u\,A^2+3\,B\,(A'+B),\\ \Pi(\lambda;x)=(3\,v-u')\,A^3+(2\,A'''+3\,B''+3\,u\,B+2\,u\,A')\,A^2+\\ +3\,A\,B\,(A''+B')+3\,B^2\,(A'+B).\end{gathered}$$

The second expression for the Ψ-function in (2) was obtained by conversion of the first expression in (2) into a polynomial as a rational function in μ with the help of the equation of the curve: $W(\mu,\lambda)\equiv\mu^3+Q(\lambda)\,\mu+R(\lambda)=0$. A sum of zeroes $\{\gamma_k\}$ on all sheets of the Riemann surface $W(\mu,\lambda)=0$ is defined by the poles of the expression Ψ'/Ψ and factorises the denominator $\Pi=a\cdot(\lambda-\gamma_1)\cdots(\lambda-\gamma_n)$. In turn, the denominator of the first formula in (2) defines the second coordinate (number of a sheet) on the curve of the Ψ-function zero: $\mu_k=\mu(\gamma_k(x))$. Thus, $F(\lambda;x)$ leads to $\mu_k(x)\,A(\gamma_k;x)$ when $\lambda\to\gamma_k$. Using this fact and taking the passage to the limit $\lambda\to\gamma_k$ in the second formula (2) one obtains analogs of the DE.

$$\gamma_k'=3\,A(\gamma_k;x)\,\frac{3\,\mu_k^2+Q(\gamma_k)}{a\prod\limits_{j\neq k}(\gamma_k-\gamma_j)},\qquad \mu_k'=-3\,A(\gamma_k;x)\,\frac{Q'(\gamma_k)\,\mu_k+R'(\gamma_k)}{a\prod\limits_{j\neq k}(\gamma_k-\gamma_j)}, \quad (3)$$

$$\mu_k=\frac{F(\gamma_k;\,x)}{A(\gamma_k;\,x)}.$$

Up to this point we have not placed any restrictions, therefore the equations (3) hold for any operator pencil (1) and the method of derivation is spread to the higher orders without any changes. The first system of equations in (3) recently appeared in [4] (not taking into account a misprint in eq. (5.34) on p. 852) as an example of the Boussinesq equation. However proofs and examples were not presented. We would like to emphasise two important circumstances not touched on in [4].

1. The DE must be reduced to an autonomous form, but in the form (3), they contains a potential in A-function;
2. The potential $[\,U\,]$ is expressed in terms of Ψ-function (i.e. Θ).

Based on this, we formulate a problem: *which operator λ-pencils allow us to recover uniquely a finite-gap potential by zeros-coordinates of the Ψ-function*? *If yes, whether it will be an Abelian function on a Jacobian of a curve*? Note, that the Schrödinger operator (excepting the simple modifications of a 2×2-Dirac-operator) is the only known example, where this question disappears due to the first trace formula containing $\sum\gamma_k$. We call it a *central* trace formula

as all others are its consequence. But an arbitrary Abelian function is a symmetrical combination of the upper bounds of the Abelian integrals in the Jacobi inverse problem. Thus, coordinates $(\gamma_k,\, \mu_k)$ must be considered as having a same importance, and possibly both will appear in the trace equalities. This requires the appearance of the second group of the equations (3). Note, that the third formula in (3) is the consequence of an algebraic curve $W(\mu,\, \lambda) = 0$ and the Ψ-function formula (2), i.e. the second coordinate μ_k is always determined. It is not difficult to find examples, when solution of the point 1) is not clear (or even impossible) and a general recipe for the solution of the problem is not known at present.

Without specification of the A, B-polynomials and the potential $[U]$, traces and the Jacobi inverse problem can not be written, because they entirely define the structure of a curve and essential singularities of the Ψ-function. Therefore, let us consider an example (genus $g = 4$).

$$\Psi''' + u(x)\,\Psi' = \lambda\,\Psi, \qquad (3\,u' - 9\,\lambda)\,\Psi'' + (u^2 - u'' + \alpha)\,\Psi' - 6\,\lambda\,u\,\Psi = \mu\,\Psi,$$

$$W(\mu,\, \lambda): \qquad \mu^3 + (27\,\alpha\,\lambda^2 + E_2)\,\mu + 729\,\lambda^5 + 81\,E_1\,\lambda^3 + E_3\,\lambda = 0. \tag{4}$$

A factorisation of the Π-polynomial yields only the formula $2\,u' = 3\sum\gamma_k$. But combining it with the expression of the integrals $E_{1,3}(u, u', \ldots, u^{(\mathrm{IV})})$ and formula (3) we obtain, as an answer to the above question, three versions of the central trace formula:

$$u = -\frac{3}{2\,\alpha}\left\{E_1 + \sum_{k=1}^{4}(\gamma_k''' + \tfrac{15}{2}\,\gamma_k^2)\right\}, \qquad u = \frac{1}{6}\sum_{k=1}^{4}\gamma_k\,\mu_k\,\frac{\sum^{4}\gamma_j - 2\,\gamma_k}{\prod\limits_{j\neq k}(\gamma_k - \gamma_j)},$$

$$u = \frac{E_3 + 3^6\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4}{6\,E_1}.$$

The integral form of equations (3) as the Jacobi inverse problem and the base of the holomorphic differentials for the trigonal curve (4) have the form

$$\left\{\begin{array}{ll} \displaystyle\sum_{k=1}^{4}\int^{(\gamma_k,\,\mu_k)}\frac{d\lambda}{3\,\mu^2 + Q(\lambda)} = a_1, & \displaystyle\sum_{k=1}^{4}\int^{(\gamma_k,\,\mu_k)}\frac{\lambda\,d\lambda}{3\,\mu^2 + Q(\lambda)} = a_2, \\ \displaystyle\sum_{k=1}^{4}\int^{(\gamma_k,\,\mu_k)}\frac{\mu\,d\lambda}{3\,\mu^2 + Q(\lambda)} = a_3, & \displaystyle\sum_{k=1}^{4}\int^{(\gamma_k,\,\mu_k)}\frac{\lambda^2\,d\lambda}{3\,\mu^2 + Q(\lambda)} = a_4 - \frac{1}{81}\,x, \end{array}\right. \tag{5}$$

where $Q(\lambda) = 27\,\alpha\,\lambda^2 + E_2$. The following nontrivial example is taken from the hierarchy of Boussinesq's equation without reduction.

$$\Psi''' + u\,\Psi' + v\,\Psi = \lambda\,\Psi, \quad u\,\Psi''' + (3\,\lambda + v - u' + \alpha)\,\Psi' + \left(\frac{2}{3}\,u'' - v' + \frac{2}{3}\,u^2\right)\Psi = \mu\,\Psi,$$

$$A = u, \qquad B = 3\,\lambda + v - u' + \alpha, \qquad \text{genus} \quad g = 3.$$

The second coordinate μ_k read as

$$\mu_k = -\frac{27\,\gamma_k^2 + 9\,(2\,v - u' + 2\,\alpha)\,\gamma_k + u\,u'' + u^3 + 3\,(\alpha + v)\,(\alpha + v - 3\,u')}{3\,u}.$$

The central trace formulas have the form:

$$\frac{9}{u} = -\sum_{k=1}^{3} \frac{\mu_k}{\prod\limits_{j\neq k}^{3} (\gamma_k - \gamma_j)}, \qquad v = \frac{2}{3}\,u' - \alpha - \sum_{k=1}^{3} \gamma_k.$$

We do not display here the correspondent curve, autonomous DE and their integral form type as (5). Note, the holomorphic Abelian differentials for an arbitrary algebraic curve $W(\mu, \lambda) = 0$ may be written in the canonical form $d\,\omega = P(\mu, \lambda)\,W_\mu^{-1}\,d\lambda$ [5, § 39], but we will not always arrive at the DE in such a way. It is enough, to mention a counterexample $\Psi'' = \lambda\,u(x)\,\Psi$, where neither the straight analogs of the DE nor trace formulas occur. But, in this example, the Ψ-function is not a classical Baker–Akhiezer function with the asymptotic behaviour $\sim \exp(k\,x)$ [3], although a commuting operator, curve and Ψ-formula are written easily.

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